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A Most General Edge Elimination Polynomial – Thickening of Edges

Christian Hoffmann*
Universität des Saarlandes, Informatik
Postfach 151150
66041 Saarbrücken, Germany
hoffmann@cs.uni-sb.de

Abstract. We consider a graph polynomial $\xi(G;x,y,z)$ introduced by Ilia Averbouch, Benny Godlin, and Johann A. Makowsky (2008). This graph polynomial simultaneously generalizes the Tutte polynomial as well as a bivariate chromatic polynomial defined by Klaus Dohmen, André Pönitz, and Peter Tittmann (2003). We derive an identity which relates the graph polynomial ξ of a thickened graph (i.e. a graph with each edge replaced by k copies of it) to ξ of the original graph. As a consequence, we observe that at every point (x,y,z), except for points lying within some set of dimension 2, evaluating ξ is #P-hard. Thus, ξ supports Johann A. Makowsky's difficult point conjecture for graph polynomials (2008).

1. Introduction

We consider the following three-variable graph polynomial, which has been introduced by Ilia Averbouch, Benny Godlin, and Johann A. Makowsky [2]:

$$\xi(G;x,y,z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A\cup B)-k_{\mathrm{cov}}(B)} \cdot y^{|A|+|B|-k_{\mathrm{cov}}(B)} \cdot z^{k_{\mathrm{cov}}(B)}, \tag{1}$$

where $G=(V,E)$ is an undirected graph. In our context, graphs are allowed to have multiple edges and self loops. Such graphs are sometimes called multigraphs [8, Section 1.10]. If A is a set of edges, we write $V(A)$ for the set of vertices incident to an edge in A. $A \sqcup B$ denotes a *vertex-disjoint* union of edge

*Address for correspondence: Universität des Saarlandes, Informatik, Postfach 151150, 66041 Saarbrücken, Germany sets A and B, $k(A \cup B)$ is the number of connected components of the graph $(V, A \cup B)$, and $k_{\mathrm{cov}}(B)$ is the number of connected components of the graph $(V(B), B)$. If G is a graph, we write $V(G)$ and $E(G)$ for the set of vertices and the set of edges of G, respectively.

The polynomial ξ simultaneously generalizes two interesting graph polynomials: the Tutte polynomial [15, 7, 16] and a bivariate chromatic polynomial $P(G; x, y)$ defined by Klaus Dohmen, André Pönitz, and Peter Tittmann [9].

The Tutte polynomial of a graph with "thickened" edges (i.e. each edge replaced by k parallel edges) evaluated at some point equals the Tutte polynomial of the original graph evaluated at another point (parallel edge reduction) [10, 14, 6]. This property can be used to prove that evaluating the Tutte polynomial is hard at almost every point [10, 4].

In Section 2 of this note we observe that edge thickening has a similar effect on ξ as on the Tutte polynomial (Theorem 3). In Section 3 we conclude that for every point $(x, y, z) \in \mathbb{Q}^3$, except on a set of dimension at most 2, it is #P-hard to evaluate ξ (i.e. to compute $\xi(G; x, y, z)$ from G) (Theorem 5). Thus, ξ supports the difficult point conjecture for graph polynomials by Johann A. Makowsky [12, Conjecture 1].

2. A Point-to-Point Reduction from Thickening

Alan D. Sokal derives short proofs of identities for the bivariate Tutte polynomial using a *multivariate* Tutte polynomial [14]. In a multivariate graph polynomial, each edge can be labeled by a different variable. The graph polynomial then depends on the actual graph, including the variables by which the edges are labeled. This approach is very effective to analyze the "behavior" of the Tutte polynomial under edge thickening: An easy calculation shows that the multivariate Tutte polynomial of a graph G with two parallel edges e_1, e_2 labeled y_1, y_2 equals the multivariate Tutte polynomial of a slightly modified graph G'. The modified graph G' is just G with e_1 and e_2 replaced by a single edge e labeled $(1+y_1)(1+y_2)-1$ [14, Section 4.4]. Once this is known, it is again easy to derive an identity describing the effect of k-thickening (i.e. replacing *every* edge by k copies of it) on the Tutte polynomial [6].

In this section we apply the above approach to ξ. Lemma 2 is the main technical contribution of this note. It describes how – in the case of polynomial ξ – two parallel edges can be simulated by a single edge. Theorem 3 describes the effect of k-thickening on ξ.

Let us introduce a different y-variable y_e for each edge e of the graph, and let $\bar{y} = (y_e)_{e \in E(G)}$. We define the following multivariate auxiliary polynomial:

$$\psi(G; x, \bar{y}, z) = \sum_{(A \sqcup B) \subseteq E(G)} w(G; x, \bar{y}, z; A, B), \tag{2}$$

where

$$w(G; x, \bar{y}, z; A, B) = x^{k(A \cup B)} \cdot \Big(\prod_{e \in (A \cup B)} y_e \Big) \cdot z^{k_{\mathrm{cov}}(B)}.$$

We write $\psi(G; x, y, z)$ for $\psi(G; x, \bar{y}, z)$ if $y_e = y$ for each $e \in E(G)$.

The following is easy to see.

Lemma 1. We have the polynomial identities

$$\psi(G; x, y, zx^{-1}y^{-1}) = \xi(G; x, y, z) \quad \text{and} \quad \xi(G; x, y, zxy) = \psi(G; x, y, z).$$

Let G be a graph, and let $e = uv \in E(G)$ be an edge. Let $E' = E(G) \setminus \{e\}$ and G_{ee} be the graph G with e doubled, i.e. $G_{ee} = (V(G), E' \cup \{e_1, e_2\})$ with $e_1 = uv$, $e_2 = uv$ being two different new edges. With respect to ψ, the relation between the original graph and the derived graph is as follows.

Lemma 2. $\psi(G_{ee}; x, \bar{y}, z) = \psi(G; x, \bar{Y}, z)$ with $Y_e = (1+y_{e_1})(1+y_{e_2})-1$ and $Y_{\tilde{e}} = y_{\tilde{e}}$ for all $\tilde{e} \in E'$.

Proof:
Let $M(G) = \{(A, B) \mid A \sqcup B \subseteq E(G)\}$ and $M(G_{ee}) = \{(\tilde{A}, \tilde{B}) \mid \tilde{A} \sqcup \tilde{B} \subseteq E(G_{ee}\}$. We define a map $\tau : M(G) \to 2^{M(G_{ee})}$ in the following way. Consider $(A, B) \in M(G)$. If $e \notin A \cup B$, we set $\tau(A, B) = \{(A, B)\}$. If $e \in A$, we let $A' := A \setminus \{e\}$ and define $\tau(A, B) = \{(A' \cup \{e_1\}, B), (A' \cup \{e_2\}, B), (A' \cup \{e_1, e_2\}, B)\}$. (In this case, $e \notin B$, as A and B are vertex-disjoint.) If $e \in B$, we let $B' := B \setminus \{e\}$ and define $\tau(A, B) = \{(A, B' \cup \{e_1\}), (A, B' \cup \{e_2\}), (A, B' \cup \{e_1, e_2\})\}$. Observe that

$$M(G_{ee}) = \bigcup_{(A,B) \in M(G)} \tau(A, B), \tag{3}$$

and that this union is a union of *pairwise disjoint* sets. Calculation yields

$$w(G; x, \bar{Y}, z; A, B) = \sum_{(\tilde{A},\tilde{B}) \in \tau(A,B)} w(G_{ee}, x, \bar{y}, z; \tilde{A}, \tilde{B}) \tag{4}$$

for every $(A, B) \in M(G)$. Thus,

$$\begin{aligned}
\psi(G_{ee}; x, \bar{y}, z) &= \sum_{(\tilde{A},\tilde{B}) \in M(G_{ee})} w(G_{ee}; x, \bar{y}, z; \tilde{A}, \tilde{B}) \\
&= \sum_{(A,B) \in M(G)} \sum_{(\tilde{A},\tilde{B}) \in \tau(A,B)} w(G_{ee}; x, \bar{y}, z; \tilde{A}, \tilde{B}) && \text{by (3)} \\
&= \sum_{(A,B) \in M(G)} w(G; x, \bar{Y}, z; A, B) && \text{by (4)} \\
&= \psi(G; x, \bar{Y}, z).
\end{aligned}$$

□

The following theorem is obtained by applying Lemma 2 repeatedly, and Lemma 1 to convert between ψ and ξ.

Theorem 3. Let G_k be the k-thickening of G (i.e. the graph obtained from G by replacing each edge e by k different copies of e). Then

$$\psi(G_k; x, y, z) = \psi(G; x, (1+y)^k - 1, z) \quad \text{and}$$
$$\xi(G_k; x, y, z) = \xi\Big(G; x, (1+y)^k - 1, z\frac{(1+y)^k - 1}{y}\Big).$$

3. Hardness

In the context of developing a general theory for graph polynomials, Johann A. Makowsky discusses the computational complexity of graph polynomials [12, Section 4]. In particular, he addresses the following question for any graph polynomial p: Fix some point x. How hard is it to compute $p(G;x)$ given G? He observes that for a number of graph polynomials this question has been answered, and they all show the following behavior: for some special points it is easy (i.e., polynomial time computable), but for "most" points it is hard (i.e. $\#\mathsf{P}$-hard). This brings him to his difficult point conjecture [12, Conjecture 1], which states that *every* extended second order logic definable graph polynomial shows this behavior. (To be precise, the difficult point conjecture also states that evaluation at a difficult point is reducible to evaluation at any other difficult point.) In this section, we show that for all points (x,y,z), except for points within some set of dimension 2, $\xi(G;x,y,z)$ is $\#\mathsf{P}$-hard to compute from G (Theorem 5).

The following theorem has been proven independently by Ilia Averbouch (Johann A. Makowsky, personal communication, October 2007), but the proof has not been published yet.

Theorem 4. Let P denote the bivariate chromatic polynomial defined by Dohmen et al. [9]. For every $(x,y) \in \mathbb{Q}$, $y \neq 0$, $(x,y) \notin \{(1,1),(2,2)\}$, it is $\#\mathsf{P}$-hard to compute $P(G;x,y)$ from G.

Proof:
Jaeger et al. [10] take the following construction from Linial [11]: Given a graph $G = (V,E)$ let $\tilde{G}$ denote the graph obtained from G by adding a new vertex $\tilde{v}$ and connecting $\tilde{v}$ to all vertices in V. Let $P(G;y)$ denote the chromatic polynomial [13]. It is not hard to see that

$$P(\tilde{G};y) = yP(G;y-1). \tag{5}$$

From this and a theorem of Dohmen et al. [9, Theorem 1] we can derive

$$P(\tilde{G};x,y) = yP(G;x-1,y-1) + (x-y)P(G;x,y). \tag{6}$$

Now we follow a standard technique for graph polynomials.

Assume that we can evaluate P at $(y+d,y)$ (i.e. compute $P(G;y+d,y)$ from G). Then we can do the following: Compute $\tilde{G}$, $P(\tilde{G};y+d,y)$ and $P(G;y+d,y)$. By (6), this gives us $P(G;y+d-1,y-1)$. Thus, we have reduced evaluation of P at $(y+d-1,y-1)$ to evaluation at $(y+d,y)$.

Let us apply this repeatedly. If y is a positive integer, we reach $(1+d,1)$. Otherwise, we can evaluate P at arbitrary many different points on the line $x = y+d$, provided that $y \neq 0$. This allows us to interpolate P on that line (the degree of $P(G;x,y)$ in x and y is at most $|V(G)|$). Thus, in both cases the ability to evaluate P at $(y+d,y)$ implies the ability to evaluate it at $(1+d,1)$.

On the line $y = 1$, polynomial P equals the independent set polynomial [9, Corollary 2]. As this graph polynomial is $\#\mathsf{P}$-hard to evaluate everywhere except at one point [3, 5], polynomial P is $\#\mathsf{P}$-hard to evaluate everywhere on the line $y = 1$, except at the point $(x,y) = (1,1)$. On the line $x = y$, polynomial P coincides with the chromatic polynomial and is thus $\#\mathsf{P}$-hard to evaluate, except at $(1,1)$ and $(2,2)$ [11, 10]. □

Theorem 5. For every $(x,y,z) \in \mathbb{Q}^3$, $x \neq 0$, $z \neq -xy$, $(x,z) \notin \{(1,0),(2,0)\}$, $y \notin \{-2,0\}$, it is $\#\mathsf{P}$-hard to compute $\xi(G;x,y,z)$ from G.

Proof:
For $x, y \in \mathbb{Q}$, $x, y \neq 0$ and $(x, y) \notin \{(1,1),(2,2)\}$, the following problem is #P-hard by Theorem 4: Given G, compute

$$P(G; x, y) = \xi(G; x, -1, x - y) = \psi\Big(G; x, -1, \frac{y - x}{x}\Big),$$

where the first equality is by Averbouch et al. [1, Proposition 18] and the second by Lemma 1. We will argue that, for any fixed $\tilde{y} \in \mathbb{Q} \setminus \{-2, 0\}$, this reduces to compute $\psi(G; x, \tilde{y}, \frac{y-x}{x})$ from G. Then we use

$$\psi\Big(G; x, \tilde{y}, \frac{y - x}{x}\Big) = \xi(G; x, \tilde{y}, (y - x)\tilde{y}),$$

which holds by Lemma 1. Finally, an easy calculation converts the conditions on $x, \tilde{y}, y$ into conditions on x, y, z and yields the theorem.

Now assume that we are able to evaluate ψ at some fixed $(x, y, z) \in \mathbb{Q}^3$, i.e. given G we can compute $\psi(G; x, y, z)$. Then Theorem 3 allows us to evaluate ψ at (x, y', z) for infinitely many different $y' = (1 + y)^k - 1$ provided that $|1 + y| \neq 0$ and $|1 + y| \neq 1$. As ψ is a polynomial, this enables interpolation in y and eventually gives us the ability to evaluate ψ at (x, y', z) for *any* $y' \in \mathbb{Q}$. In particular, being able to evaluate ψ at $(x, \tilde{y}, \frac{y-x}{x})$, $\tilde{y} \in \mathbb{Q} \setminus \{-2, 0\}$, implies the ability to evaluate it at $(x, -1, \frac{y-x}{x})$. □

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